

GENERALIZATIONS OF WEI'S DUALITY THEOREM

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ABSTRACT. Wei's celebrated Duality Theorem is generalized in several ways, expressed as duality theorems for linear codes over division rings and, more generally, duality theorems for matroids. These results are further generalized, resulting in two Wei-type duality theorems for new combinatorial structures that are introduced and named *demi-matroids*. These generalize matroids and are the appropriate combinatorial objects for describing the duality in Wei's Duality Theorem. A new proof of the Duality Theorem is thereby given that explains the theorem in combinatorial terms. Special cases of the general duality theorems are also given, including duality theorems for cycles and bonds in graphs and for transversals.

1. INTRODUCTION

In 1991, Victor K. Wei presented his remarkably simple and elegant Duality Theorem [1, Theorem 3] concerning the generalized Hamming weights (higher weights) of a code over a finite field and of its dual code. This result has attracted much favorable attention among coding theoreticians and has created an active sub-field of research, while also re-invigorating the studies of higher weight enumerators (support weight enumerators) for linear codes over finite fields (see [2, 3, 4, 5, 6, 7, 8, 9, 10] for instance).

Wei's Duality Theorem has been re-proved and generalised by several authors, often with respect to codes over certain finite rings (see [11, 12, 13, 14] for a few generalizations). In the present paper, the Duality Theorem is generalized with respect to matroids, and more generally, to demi-matroids. Matroids capture the combinatorial essence of linear independence, and recent research has demonstrated how matroid theory may be applied with good effect to coding-theoretical problems (see [15, 16]). More general than matroids are the demi-matroids, which turn out to be the appropriate combinatorial structures for describing the duality in Wei's Duality Theorem. Section 2 introduces demi-matroids and describes some of their parameters and properties. In addition to the matroid dual, each demi-matroid also has another type of dual, its *supplement*. The dual and the supplement each give rise to a generalization of Wei's Duality Theorem and thereby explain in combinatorial terms why Wei's Duality Theorem holds.

Special cases of these generalizations are presented in Section 3. The first special case is Wei's Duality Theorem generalized with respect to matroids. In turn, this result has as special cases two duality results, namely for graphs (first proved in [16]) and for transversals. In each of these two cases, the matroid-analogues of the

Date: October 12, 2009.

1991 *Mathematics Subject Classification.* 05B35.

T. Britz was supported by an ARC Discovery Grant.

generalized Hamming weights have natural graph- or transversal interpretations. A final special case is a duality theorem concerning the generalized Hamming weights of a code over a division ring and of its dual code; this generalizes Wei's Duality Theorem and provides a new proof thereof.

This paper is largely self-explanatory but does at times refer to some elementary matroid theory. For information on matroids, see the excellent expositions [17, 18].

2. WEI-TYPE DUALITY THEOREMS FOR DEMI-MATROIDS

A *demi-matroid* is a triple (E, s, t) consisting of a set E and two functions $s, t : 2^E \mapsto \mathbb{N}_0$ satisfying the following two conditions for all subsets $X \subseteq Y \subseteq E$:

- (R) $0 \leq s(X) \leq s(Y) \leq |Y|$ and $0 \leq t(X) \leq t(Y) \leq |Y|$;
- (D) $|E - X| - s(E - X) = t(E) - t(X)$.

Note that $s(\emptyset) = t(\emptyset) = 0$ by (R). It follows that (D) is equivalent to the following condition:

- (D') $|E - X| - t(E - X) = s(E) - s(X)$.

Note that for any matroid M on E with rank function ρ , the triple (E, ρ, ρ^*) is a demi-matroid. Conversely, if (E, s, t) is a demi-matroid, then s is the rank function of a matroid M on E if and only if t is the rank function of M^* . The following example shows that demi-matroids properly generalize matroids.

Example 1. Suppose that $E = \{a, b\}$ and define $s(X) := 0$ for $X = \emptyset, \{a\}, \{b\}$, and $s(E) := 1$. The triple (E, s, s) is a demi-matroid but s is not the rank function of any (poly)matroid on E .

Let E be a set of n elements, and let $D = (E, s, t)$ be a demi-matroid. By (D),

$$s(E) + t(E) = n.$$

Set $k := s(E)$.

Lemma 2. $s(X - x) \geq s(X) - 1$ and $t(X - x) \geq t(X) - 1$ for all $X \subseteq E$ and $x \in E$.

Proof. By (R) and (D),

$$\begin{aligned} t(X - x) &= t(E) - |E - (X - x)| + s(E - (X - x)) \\ &\geq t(E) - |E - X| - 1 + s(E - X) \\ &= t(X) - 1. \end{aligned}$$

Similarly, $s(X - x) \geq s(X) - 1$. □

Define for all $i = 0, \dots, k$ and $j = 0, \dots, n - k$,

$$\begin{aligned} \sigma_i &:= \min\{|X| : X \subseteq E, s(X) \geq i\}; \\ \tau_j &:= \min\{|X| : X \subseteq E, t(X) \geq j\}; \\ s_i &:= \max\{|X| : X \subseteq E, s(X) \leq i\}; \\ t_j &:= \max\{|X| : X \subseteq E, t(X) \leq j\}. \end{aligned}$$

By (R) and Lemma 2, all of the numbers σ_i , τ_i , s_i , and t_j are well-defined and may be given the following equivalent characterizations:

Lemma 3. For all $i = 0, \dots, k$ and $j = 0, \dots, n - k$,

$$\begin{aligned}\sigma_i &= \min\{|X| : X \subseteq E, s(X) = i\}; \\ \tau_j &= \min\{|X| : X \subseteq E, t(X) = j\}; \\ s_i &= \max\{|X| : X \subseteq E, s(X) = i\}; \\ t_j &= \max\{|X| : X \subseteq E, t(X) = j\}.\end{aligned}$$

Remark 4. If M is a matroid on E with rank function ρ , then the coefficients σ_i, τ_j for demi-matroid $D := (E, \rho, \rho^*)$ are trivial: $\sigma_i = i$ and $\tau_j = j$ for all i, j .

Lemma 5. The following inequalities hold:

$$\begin{aligned}0 &= \sigma_0 < \sigma_1 < \sigma_2 < \dots < \sigma_k \leq n; \\ 0 &= \tau_0 < \tau_1 < \tau_2 < \dots < \tau_{n-k} \leq n.\end{aligned}$$

Proof. For each $i = 1, \dots, k$, let $X \subseteq E$ be a subset such that $|X| = \sigma_i$ and $s(X) \geq i$. By Lemma 2, $s(X - x) \geq i - 1$ for any $x \in X$, so $\sigma_{i-1} \leq |X - x| < \sigma_i$.

Similarly, $\tau_{j-1} < \tau_j$ for each $j = 1, \dots, n - k$. \square

Lemma 6. The following inequalities hold:

$$\begin{aligned}0 &\leq s_0 < s_1 < s_2 < \dots < s_k = n; \\ 0 &\leq t_0 < t_1 < t_2 < \dots < t_{n-k} = n.\end{aligned}$$

Proof. Similar to the proof of Lemma 5. \square

The four above monotonicities each induce a generalized Singleton-type bound for demi-matroids:

Corollary 7. For all $i = 0, \dots, k$ and $j = 0, \dots, n - k$,

$$\begin{aligned}\sigma_i &\leq n - k + i; \\ s_i &\leq n - k + i; \\ \tau_j &\leq k + j; \\ t_j &\leq k + j.\end{aligned}$$

The *dual demi-matroid* of a demi-matroid $D := (E, s, t)$ is the triple $D^* := (E, t, s)$. The operation $D \mapsto D^*$ is clearly an involution, i.e.,

$$D = (D^*)^*.$$

A second fundamental involution on demi-matroids is now presented. For any real function $f : 2^E \mapsto \mathbb{R}$, let \bar{f} denote the function given by

$$\bar{f}(X) := f(E) - f(E - X).$$

Since

$$\bar{\bar{f}}(X) = \bar{f}(E) - \bar{f}(E - X) = f(X) - f(\emptyset),$$

it follows that if $f(\emptyset) = 0$, then the operation $f \mapsto \bar{f}$ is an involution, i.e., $f = \bar{\bar{f}}$.

Theorem 8. The triple $\bar{D} := (E, \bar{s}, \bar{t})$ is a demi-matroid; furthermore, $D = \bar{\bar{D}}$ and $\bar{D}^* = \bar{D}^*$.

Proof. To show that \overline{D} is a demi-matroid, first note that $\overline{s}(\emptyset) = \overline{t}(\emptyset) = 0$ and that $\overline{s}(E) = s(E)$ and $\overline{t}(E) = t(E)$. Consider subsets $X \subseteq Y \subseteq E$. By (R) and (D),

$$0 \leq s(E) - s(E - X) \leq s(E) - s(E - Y) = |Y| - t(Y) \leq |Y|,$$

so $0 \leq \overline{s}(X) \leq \overline{s}(Y) \leq |Y|$. Similarly, it is easy to show that $0 \leq \overline{t}(X) \leq \overline{t}(Y) \leq |Y|$, so \overline{D} satisfies (R). By (D'),

$$\begin{aligned} |E - X| - \overline{s}(E - X) &= |E - X| - (s(E) - s(X)) \\ &= t(E - X) \\ &= t(E) - \overline{t}(X) \\ &= \overline{t}(E) - \overline{t}(X), \end{aligned}$$

so \overline{D} satisfies (D). Hence, \overline{D} is a demi-matroid.

Finally, note that $D = (E, s, t) = (E, \overline{s}, \overline{t}) = \overline{\overline{D}}$ and that $\overline{D^*} = \overline{(E, t, s)} = (E, \overline{t}, \overline{s}) = (E, \overline{s}, \overline{t})^* = \overline{D^*}$. \square

The demi-matroid \overline{D} is called the *supplement* of D .

Example 9. The supplement operation does not generally apply to matroids. For instance, consider the matroid $M := (E, \rho)$ where $E = \{a, b, c\}$ and $\rho(X) = 0$ for $X = \emptyset, \{a\}$ and $\rho(X) = 1$ for all other subsets $X \subseteq E$. Then $D := (E, \rho, \rho^*)$ is a demi-matroid, so $\overline{D} = (E, \overline{\rho}, \overline{\rho^*})$ is also a demi-matroid. However, $(E, \overline{\rho})$ is not a matroid, since it would have rank 1 but only contain loops.

Define for all $i = 0, \dots, k$ and $j = 0, \dots, n - k$,

$$\begin{aligned} \overline{\sigma}_i &:= \min\{|X| : X \subseteq E, \overline{s}(X) = i\}; \\ \overline{\tau}_j &:= \min\{|X| : X \subseteq E, \overline{t}(X) = j\}; \\ \overline{s}_i &:= \max\{|X| : X \subseteq E, \overline{s}(X) = i\}; \\ \overline{t}_j &:= \max\{|X| : X \subseteq E, \overline{t}(X) = j\}. \end{aligned}$$

Lemma 10. For each $i = 0, \dots, k$ and $j = 0, \dots, n - k$,

$$\begin{aligned} s_i &= n - \overline{\sigma}_{k-i}; \\ \sigma_i &= n - \overline{s}_{k-i}; \\ t_j &= n - \overline{\tau}_{n-k-j}; \\ \tau_j &= n - \overline{t}_{n-k-j}. \end{aligned}$$

Proof. By Lemma 3,

$$\begin{aligned} s_i &= \max\{|X| : X \subseteq E, s(X) = i\} \\ &= \max\{|E - X| : X \subseteq E, s(E - X) = i\} \\ &= n - \min\{|X| : X \subseteq E, \overline{s}(X) = k - i\} \\ &= n - \overline{\sigma}_{k-i}. \end{aligned}$$

The remaining identities are proved similarly. \square

For each demi-matroid D , set

$$\begin{aligned} S_D &:= \{n - s_{k-1}, \dots, n - s_1, n - s_0\}; \\ T_D &:= \{t_0 + 1, t_1 + 1, \dots, t_{n-k-1} + 1\}; \\ U_D &:= \{\sigma_1, \sigma_2, \dots, \sigma_k\}; \\ V_D &:= \{n + 1 - \tau_{n-k}, \dots, n + 1 - \tau_2, n + 1 - \tau_1\}. \end{aligned}$$

Lemma 10 implies the following identities:

Lemma 11. $S_D = U_{\overline{D}}$ and $T_D = V_{\overline{D}}$.

The main results of this paper are the following fundamental duality theorems for demi-matroids that each generalize Wei's Duality Theorem [1].

Theorem 12. $S_D \cup T_D = \{1, \dots, n\}$ and $S_D \cap T_D = \emptyset$.

Theorem 13. $U_D \cup V_D = \{1, \dots, n\}$ and $U_D \cap V_D = \emptyset$.

Proof of Theorems 12 and 13. Assume that there are integers i, j such that $\sigma_i = n + 1 - \tau_j$. Let $X \subseteq E$ be a subset satisfying $|X| = \tau_j$ and $t(X) \geq j$. Then $|E - X| = \sigma_i - 1$, so $s(E - X) \leq i - 1$ from Lemma 5. By (D),

$$n - \tau_j - (n - k) + j = -\tau_j + k + j \leq i - 1.$$

Similarly,

$$n - \sigma_i - k + i \leq j - 1.$$

Hence, $-1 = n - \sigma_i - \tau_j \leq -2$, a contradiction. This proves Theorem 13.

To prove Theorem 12, apply Theorem 13 to \overline{D} , and use Lemma 11. \square

Example 14. For the demi-matroid $D := (E, s, t)$ with $E = \{a, b, c\}$, $s(E) = 1$, and $s(X) = 0$ for $X \subsetneq E$,

$$\begin{aligned} S_D &= \{1\} & U_D &= \{3\}; \\ T_D &= \{2, 3\} & V_D &= \{1, 2\}. \end{aligned}$$

Thus, $S_D \cup T_D = \{1, 2, 3\}$ and $S_D \cap T_D = \emptyset$, as asserted by Theorem 12. Similarly, $U_D \cup V_D = \{1, 2, 3\}$ and $U_D \cap V_D = \emptyset$, as asserted by Theorem 13.

3. DUALITY THEOREMS FOR MATROIDS, GRAPHS, TRANSVERSALS, AND LINEAR CODES

Let $M = (E, \rho)$ be a matroid of rank $k := \rho(M)$ on the set E . For all $i = 0, \dots, k$ and $j = 0, \dots, n - k$, define

$$\begin{aligned} f_i &:= \max\{|F| : F \subseteq E, \rho(F) = i\}; \\ f_j^* &:= \max\{|F| : F \subseteq E, \rho^*(F) = j\}. \end{aligned}$$

Set

$$\begin{aligned} S_M &:= \{n - f_{k-1}, \dots, n - f_1, n - f_0\}; \\ T_M &:= \{f_0^* + 1, f_1^* + 1, \dots, f_{n-k-1}^* + 1\}. \end{aligned}$$

The following duality result for matroids follows immediately from Theorem 13.

Theorem 15. $S_M \cup T_M = \{1, \dots, n\}$ and $S_M \cap T_M = \emptyset$.

Example 16. The non-representable *Vámos matroid* V_8 on $E := \{1, \dots, 8\}$ has as its bases $\mathcal{B}(M)$ all 4-subsets of E except for the following:

$$\{1, 2, 5, 6\}, \{1, 3, 5, 7\}, \{1, 4, 5, 8\}, \{2, 3, 6, 7\}, \{2, 4, 6, 8\}.$$

The matroid $M := V_8$ is simple, self-dual, and paving, so

$$(f_0, f_1, f_2, f_3) = (f_0^*, f_1^*, f_2^*, f_3^*) = (0, 1, 2, 4).$$

It follows that $S_M = \{4, 6, 7, 8\}$ and $T_M = \{1, 2, 3, 5\}$. Thus, $S_M \cap T_M = \emptyset$ and $S_M \cup T_M = \{1, \dots, 8\}$, as asserted by Theorem 15.

Remark 17. The demi-matroid $D := (E, \rho, \rho^*)$ satisfies $\sigma_i = i$ and $\tau_j = j$ for all $i = 0, \dots, k$ and $j = 0, \dots, n - k$. Thus, $U_D = \{1, 2, \dots, k\}$ and $V_D = \{k + 1, \dots, n - 1, n\}$. Hence, there is no interesting matroid analogue of Theorem 12.

The coefficients f_i and f_j^* can fairly easily be re-expressed in terms of cocircuits and circuits (see [19, p. 306]):

$$(F) \quad \begin{aligned} n - f_{i-1} &= \min\{|X| : X = \bigcup_{j=1}^i B_j \text{ where, for all } j \leq i, \\ &B_j \in \mathcal{C}^*(M), B_j \not\subseteq \bigcup_{k \neq j} B_k\}; \end{aligned}$$

$$(F^*) \quad \begin{aligned} n - f_{j-1}^* &= \min\{|X| : X = \bigcup_{i=1}^j C_i \text{ where, for all } i \leq j, \\ &C_i \in \mathcal{C}(M), C_i \not\subseteq \bigcup_{k \neq i} C_k\}. \end{aligned}$$

These identities will be used in the subsections below.

3.1. Perfect matroid designs. A *perfect matroid design* is a matroid M in which the cardinality of each closed set is determined uniquely by its rank (see [18, Chapter 12]). If the rank of a closed set F of M is i , then $|F| = f_i$. Theorem 15 immediately implies the following result.

Corollary 18. *The cardinalities of the closed sets of a perfect matroid design M are uniquely determined by the closed set cardinalities of M^* .*

3.2. Graphs. Let G be a (multi)graph on n edges whose spanning forests each contains k edges. Recall that a *bond* of G is a minimal cut-set of edges of G . For each $i = 1, \dots, k$ and $j = 1, \dots, n - k$, define

$$\begin{aligned} b_i &:= \text{minimal number of edges in a union of } i \text{ bonds,} \\ &\quad \text{none contained in the union of the others;} \\ c_j &:= \text{minimal number of edges in a union of } j \text{ cycles,} \\ &\quad \text{none contained in the union of the others.} \end{aligned}$$

Consider the cycle matroid $M := M(G)$ and its coefficients f_i and f_j^* . Equations (F) and (F*) immediately imply the following result:

Proposition 19. $b_i := n - f_{i-1}$ and $c_j := n - f_{j-1}^*$.

Corollary 20. *The maximal number of edges in any subgraph of G whose spanning forests each contain $i - 1$ edges is $n - b_i$. Similarly, $n - c_j$ is the maximal size of an edge set $E' \subseteq E(G)$ for which $G \setminus E''$ does not span G for any j -element subset $E'' \subseteq E'$.*

Set

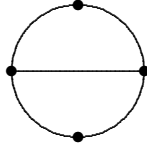
$$U_G := \{b_1, \dots, b_k\};$$

$$V_G := \{n + 1 - c_{n-k}, \dots, n + 1 - c_1\}.$$

The next result was proved in [16] and follows immediately from Theorem 15 and Proposition 19.

Theorem 21. $U_G \cup V_G = \{1, \dots, n\}$ and $U_G \cap V_G = \emptyset$.

Example 22. The graph G below has $n = 5$ edges and each of its spanning forests has $k = 3$ edges:



For this graph, $(b_1, b_2, b_3) = (2, 4, 5)$ and $(c_1, c_2) = (3, 5)$. Set

$$U_G := \{b_1, b_2, b_3\} = \{2, 4, 5\};$$

$$V_G := \{n + 1 - c_2, n + 1 - c_1\} = \{1, 3\}.$$

Then $U_G \cup V_G = \{1, 2, 3, 4, 5\}$ and $U_G \cap V_G = \emptyset$, as asserted by Theorem 21.

Example 23. For the complete graph $G := K_m$ on m vertices, the number of edges is $n = \binom{m}{2}$, and each spanning tree contains $k = m - 1$ edges. In [12], it was shown that

$$\{b_1, \dots, b_k\} = \{n - \binom{i}{2} : i = 1, \dots, k\};$$

$$\{c_1, \dots, c_{n-k}\} = \{1, \dots, n\} \setminus \left\{ \binom{i}{2} + 1 : i = 1, \dots, k \right\},$$

respectively, so $U_G \cup V_G = \{1, \dots, n\}$ and $U_G \cap V_G = \emptyset$, as asserted by Theorem 21.

Example 24. For the complete bipartite graph $G := K_{l,m}$ with $l \geq m$, the number of edges is $n = lm$, and each spanning tree contains $k = l + m - 1$ edges. In [12], it was shown that

$$\{b_1, \dots, b_k\} = \{m, 2m, \dots, (l - m)m\} \cup$$

$$\left\{ n - \left\lfloor \frac{i^2}{4} \right\rfloor : i = 1, \dots, 2m - 1 \right\};$$

$$\{c_1, \dots, c_{n-k}\} = \{1, \dots, n\} \setminus$$

$$\left(\{n + 1 - im : i = 1, \dots, l - m\} \cup \right.$$

$$\left. \left\{ \left\lfloor \frac{i^2}{4} \right\rfloor + 1 : i = 1, \dots, 2m - 1 \right\} \right),$$

respectively, so $U_G \cup V_G = \{1, \dots, n\}$ and $U_G \cap V_G = \emptyset$, as asserted by Theorem 21.

3.3. Transversals. Let $\mathcal{A} := \{A_1, \dots, A_m\}$ be a multiset of subsets $A_j \subseteq E$. A *transversal* of \mathcal{A} is a set $T \subseteq E$ of size $|T| = |\mathcal{A}|$ for which the elements of T may be labeled e_1, \dots, e_m so that $e_j \in A_j$ for each $j = 1, \dots, m$. A *partial transversal* of \mathcal{A} is a transversal of a sub-multiset of \mathcal{A} . The partial transversals of \mathcal{A} form the independent sets of the *transversal matroid* of \mathcal{A} , denoted by $M[\mathcal{A}]$ (cf. [17, Section 1.6]). A set $X \subseteq E$ is a *plug* for \mathcal{A} if $X - e$ is a partial transversal of \mathcal{A} for each $e \in X$ but X itself is not. Let k denote the maximal size of a partial transversal of \mathcal{A} , that is, the rank of $M[\mathcal{A}]$. For each $i = 0, \dots, k-1$ and $j = 1, \dots, n-k$, define

$$m_i := \max\{|X| : X \subseteq E, X \text{ contains a partial transversal of } \mathcal{A} \text{ of size } i \text{ but none of size } i+1\};$$

$$p_j := \text{minimal size of a union of } j \text{ plugs for } \mathcal{A}, \\ \text{none contained in the union of the others.}$$

Set

$$U_{\mathcal{A}} := \{m_0 + 1, \dots, m_{k-1} + 1\}; \\ V_{\mathcal{A}} := \{p_1, \dots, p_{n-k}\}.$$

Theorem 25. $U_{\mathcal{A}} \cup V_{\mathcal{A}} = \{1, \dots, n\}$ and $U_{\mathcal{A}} \cap V_{\mathcal{A}} = \emptyset$.

Proof. For $M := M[\mathcal{A}]$, $m_i = f_i$ and $p_j = n - f_{j-1}^*$, by (F^*) . Apply Theorem 15. \square

Example 26. Let $E := \{a, b, c, d, e\}$ and

$$\mathcal{A} := \{\{a, b\}, \{a, c\}, \{d\}, \{d\}\}.$$

Then $U_{\mathcal{A}} = \{2, 4, 5\}$ and $V_{\mathcal{A}} = \{1, 3\}$, so $U_{\mathcal{A}} \cap V_{\mathcal{A}} = \emptyset$ and $U_{\mathcal{A}} \cup V_{\mathcal{A}} = \{1, 2, 3, 4, 5\}$, as claimed by Theorem 25.

3.4. Codes over division rings. Let R denote a division ring (perhaps a field) and set $E := \{1, 2, \dots, n\}$. The *support* of each codeword $\mathbf{x} := (x_1, x_2, \dots, x_n) \in R^n$ is the set

$$\text{supp}(\mathbf{x}) := \{i : x_i \neq 0\}.$$

Similarly, the *support* and *weight* of each subset $D \subseteq R^n$ are defined as follows:

$$\text{Supp}(D) := \bigcup_{\mathbf{x} \in D} \text{supp}(\mathbf{x}); \\ \text{wt}(D) := |\text{Supp}(D)|.$$

Let C be a right linear $[n, k]$ code over R with coordinates E . The *dual code* C^\perp is given as follows:

$$C^\perp = \{\mathbf{y} \in R^n : \mathbf{x} \cdot \mathbf{y} = 0, \forall \mathbf{x} \in C\}.$$

For each integer $i = 1, \dots, k$ ($j = 1, \dots, n-k$), define the i th (j th) *generalized Hamming weight* of C (C^\perp) as follows:

$$d_i := \min\{\text{wt}(D) : D \text{ a right linear } [n, i] \text{ subcode of } C\}; \\ d_j^\perp := \min\{\text{wt}(D) : D \text{ a right linear } [n, j] \text{ subcode of } C^\perp\}.$$

For any subset $X \subseteq E$, the *punctured code* $C \setminus X$ is the right linear code obtained by deleting the coordinates X from each codeword of C . Also, $C(X)$ is the right linear subcode of C consisting of all codewords $\mathbf{x} \in C$ with $\text{supp}(\mathbf{x}) \subseteq X$. Note that $k = \dim C = \dim C \setminus X + \dim C(X)$.

Define the function $\rho_C : 2^E \mapsto \mathbb{N}_0$ by

$$\rho_C(X) := \dim C \setminus (E - X).$$

This is the rank function of the vector matroid $M_C = (E, \rho_C)$. Define ρ_{C^\perp} similarly for C^\perp and note that $\rho_C^* = \rho_{C^\perp}$. Hence,

Theorem 27. $D_C := (E, \rho_C, \rho_{C^\perp})$ is a demi-matroid.

Consider the numbers σ_i and τ_j for D_C .

Proposition 28. *The following identities hold:*

$$d_i = \bar{\sigma}_i = n - s_{k-i};$$

$$d_j^\perp = \bar{\tau}_j = n - t_{n-k-j}.$$

Proof. Let D be a right linear $[n, i]$ subcode of C with $\text{wt}(D) = d_i$, and set $X = \text{Supp}(D)$. Then

$$\bar{\rho}_C(X) = k - \rho_C(E - X) \geq \dim D = i,$$

so $d_i = |X| \geq \bar{\sigma}_i$.

Conversely, let $X \subseteq E$ be a subset with $\bar{\rho}_C(X) = i$ and $|X| = \bar{\sigma}_i$. Then $\rho_C(E - X) = k - i$. Now, $C(X)$ is a right linear subcode of C with $\dim C(X) = k - \rho_C(E - X) = i$ and $\text{wt}(C(X)) \leq |X|$. Hence, $d_i \leq |X| = \bar{\sigma}_i$.

It follows that $d_i = \bar{\sigma}_i$. Similarly, $d_j^\perp = \bar{\tau}_j$, and Lemma 10 concludes the proof. \square

Set

$$U_C := \{d_1, \dots, d_k\};$$

$$V_C := \{n + 1 - d_{n-k}^\perp, \dots, n + 1 - d_1^\perp\}.$$

The result below generalizes Wei's Duality Theorem; the former specializes to the latter when R is a finite field.

Theorem 29. $U_C \cup V_C = \{1, \dots, n\}$ and $U_C \cap V_C = \emptyset$.

Proof. The theorem follows from Theorems 13 and 27 and Proposition 28. \square

Example 30. Consider the linear code C generated by the following binary matrix:

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

In this case, $R = \text{GF}(2)$, $E = \{1, 2, 3, 4, 5\}$, $n = 5$, and $k = 3$. Furthermore, $(d_1, d_2, d_3) = (2, 3, 5)$ and $(d_1^\perp, d_2^\perp) = (2, 5)$, so $U_C = \{2, 3, 5\}$ and $V_C = \{1, 4\}$. Then $U_C \cap V_C = \emptyset$ and $U_C \cup V_C = \{1, 2, 3, 4, 5\}$, as asserted by Theorem 29.

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